SIMPLE ALGEBRAS OF WEYL TYPE [1]

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Abstract Over a field $\mathbb{F}$ of any characteristic, for a commutative associative algebra A with an identity element and for the polynomial algebra $\mathbb{F}[D]$ of a commutative derivation subalgebra D of A, the associative and the Lie algebras of Weyl type on the same vector space $A[D] = A \otimes \mathbb{F}[D]$ are defined. It is proved that $A[D]$, as a Lie algebra (modular its center) or as an associative algebra, is simple if and only if A is D-simple and $A[D]$ acts faithfully on A. Thus a lot of simple algebras are obtained.

Keywords: Simple Lie algebra, simple associative algebra, derivation.

For a long time, simple Lie algebras and simple associative algebras have been two central objects in the theory of algebra. The four well-known series of infinite dimensional simple Lie algebras of Cartan type have played important roles in the structure theory of Lie algebras. Generalizations of the simple Lie algebras of Cartan type over a field of characteristic zero have been obtained by Kawamoto [1], Osborn [2], Dokovic and Zhao [3,4,5], Osborn and Zhao [6,7] and Zhao [8]. Passman [9], Jordan [10] studied the Lie algebras $AD = A \otimes D$ of generalized Witt type constructed from a commutative associative algebra A with an identity element and its commutative derivation subalgebra D over a field $\mathbb{F}$ of arbitrary characteristic. Passman proved that AD is simple if and only if A is D-simple and AD acts faithfully on A. Xu [11] studied some of these simple Lie algebras of Witt type and other Cartan types Lie algebras, based on the pairs of the tensor algebra of the group algebra of an additive subgroup of $\mathbb{F}^n$ with the polynomial algebra in several variables and the subalgebra of commuting locally finite derivations. Su, Xu and Zhang [12] gave the structure spaces of the generalized simple Lie algebras of Witt type constructed in [11]. We[3] determined the second cohomology group and gave some representations of the Lie algebras of generalized Witt type which are some Lie algebras defined by Passman, more general than those defined by Dokovic and Zhao, and slightly more general than those defined by Xu.

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Throughout this paper, $I\!\!F$ is a field of any characteristic.

In this paper, for a commutative associative algebra A with an identity element over $I\!\!F$ and for the polynomial algebra of its commutative derivation subalgebra D, we construct the associative algebras $A \otimes I\!\!F[D]$ of Weyl type and determine the necessary and sufficient conditions for $A \otimes I\!\!F[D]$ to be simple, as a Lie algebra and as associative algebra respectively, see Theorems 1.1, 1.2. Using this construction, by giving specific A and D, we then obtain a large class of simple algebras. Another interesting observation is that all Lie algebras mentioned above are Lie subalgebras of the Weyl type Lie algebras constructed in this paper.

1 Simple algebras of Weyl type

Let A be an commutative associative algebra with an identity element 1 over $I\!\!F$, let D be a nonzero $I\!\!F$-vector space spanned by some commuting $I\!\!F$-derivations of A. Let

$$\{\partial_i \,|\, i \in I\}, \text{ where } I \text{ is some indexing set,} \tag{1}$$

be an $I\!\!F$-basis of D. Choose a total ordering on I. Denote

$$J = \{\alpha = (\alpha_i \,|\, i \in I) \,|\, \alpha_i \in Z\!\!\!Z_+, \forall\, i \in I \text{ and } \alpha_i = 0 \text{ for all but a finite number of } i \in I\}, \tag{2}$$

where $\alpha = (\alpha_i \,|\, i \in I)$, we shall simply write $\alpha = (\alpha_i)$, is a collection of nonnegative integers indexed by I. Define a total ordering on J by

$$\alpha < \beta \Leftrightarrow |\alpha| < |\beta| \text{ or } |\alpha| = |\beta| \text{ but there } \ \exists\, i \in I, \alpha_i < \beta_i \text{ and } \alpha_j = \beta_j, \forall\, j \in I, j < i, \tag{3}$$

where $|\alpha| = \sum_{i \in I} \alpha_i$ is called the *level* of α. Denote by $I\!\!F[D]$ the polynomial algebra (the group algebra) of D with basis

$$B = \{\partial^{(\alpha)} = \prod_{i \in I} \partial_i^{\alpha_i} \,|\, \alpha \in J\}. \tag{4}$$

Let $A[D] = A \otimes I\!\!F[D]$ be the tensor product of A and $I\!\!F[D]$, which acts naturally on A by

$$u \otimes \partial^{(\alpha)} : x \mapsto u \cdot \partial^{(\alpha)}(x) = u \cdot (\prod_{i \in I} \partial_i^{\alpha_i})(x), \ \forall\, u, x \in A, \ \partial^{(\alpha)} \in B, \tag{5}$$

where $(\partial_1 \partial_2 \cdots \partial_n)(x) = (\partial_1(\partial_2 \cdots (\partial_n(x)) \cdots))$. This gives rise to a linear transformation

$$\theta : A[D] \to \mathrm{Hom}_{I\!\!F}(A, A). \tag{6}$$

For any $\alpha \in J$, set

$$\begin{aligned} &\mathrm{supp}(\alpha) = \{i \in I \,|\, \alpha_i \neq 0\}, \\ &J(\alpha) = \{\gamma \in J \,|\, \gamma_i \leq \alpha_i, \ \forall\, i \in I\}, \\ &\tbinom{\alpha}{\gamma} = \prod_{i \in I} \tbinom{\alpha_i}{\gamma_i}, \ \gamma \in J(\alpha). \end{aligned} \tag{7}$$

To obtain an associative algebra structure on $A[D]$ so that θ is a homomorphism of associative algebras, we define the product as follows

$$(u \otimes \partial^{(\alpha)}) \cdot (v \otimes \partial^{(\beta)}) = u \sum_{\gamma \in J(\alpha)} \binom{\alpha}{\gamma} \partial^{(\gamma)}(v) \otimes \partial^{(\alpha+\beta-\gamma)}, \tag{8}$$

for all $u, v \in A$, $\alpha, \beta \in J$. Then (6) defines A as an $A[D]$-module. We call the associative algebra $(A[D], \cdot)$ an *associative algebra of Weyl type*. For Weyl algebras we refer the reader to [13]. We shall simply denote $u \otimes d$ by ud for $u \in A, d \in I\!\!F[D]$. For any

$$x = \sum_{\alpha \in J} u_\alpha \partial^{(\alpha)} \in A[D], \tag{9}$$

we say x has *leading term* $ld(x) = u_\beta \partial^{(\beta)}$, *leading degree* $deg(x) = \beta$ and *leading level* $lev(x) = |\beta|$ if $u_\beta \neq 0$ and for all $\alpha \in J, u_\alpha \neq 0 \Rightarrow \alpha \leq \beta$. Define $lev(0) = -\infty$. We define the *support* of x to be the set $\{\alpha \in J \mid u_\alpha \neq 0\}$. Set

$$I\!\!F_1 = \{u \in A \mid D(u) = 0\}. \tag{10}$$

We use this notation because it is a field extension of $I\!\!F$ when A is *D-simple*, i.e., A has no nontrivial D-stable ideals, see [9].

Define the binary operation $[\cdot,\cdot]$ to be the usual induced Lie bracket on the associative algebra $(A[D], \cdot)$ so that $(A[D], [\cdot,\cdot])$ is a Lie algebra. Obviously, $I\!\!F_1$ is contained in the center of $(A[D], [\cdot,\cdot])$. Denote

$$\overline{A}[D] = A[D]/I\!\!F_1, \tag{11}$$

whose induced Lie bracket is also denoted by $[\cdot,\cdot]$. We call the Lie algebra $\overline{A}[D]$ a *Lie algebra of Weyl type*.

Theorem 1.1 The Lie algebra $\overline{A}[D]$ is simple if and only if A is D-simple and $I\!\!F_1[D]$ acts faithfully on A.

Proof. "$\Rightarrow$": If $\mathcal{I}$ is a D-stable ideal of A, then clearly $\mathcal{I}[D]$ is a Lie ideal of $A[D]$. If $\mathcal{I} \neq 0$, then $\mathcal{I}[D] \not\subset I\!\!F_1$, and so $\mathcal{I}[D] = A[D]$. Thus $\mathcal{I}$ must be A. Furthermore, the kernel kerθ of the Lie homomorphism θ is a Lie ideal of $A[D]$. If ker$\theta = A[D]$, then in particular, D acts trivially on A, contradicting its definition as a nonzero subspace of $\mathrm{Der}_{I\!\!F}(A)$. Thus ker$\theta \subset I\!\!F_1$. If $\theta(f) = 0$ for $f \in A$, then $0 = \theta(f)(1) = f$. Hence, ker$\theta = 0$, i.e., $A[D]$ acts faithfully on A. In particular, $I\!\!F_1[D]$ acts faithfully on A.

"$\Leftarrow$": We prove the sufficient conditions by several claims.

Claim 1. $A[D]$ acts faithfully on A.

Suppose ker$\theta \neq 0$. Let $m \geq 1$ be the minimal support size of nonzero elements in kerθ, i.e., there exist $I\!\!F$-linearly independent elements $d_1, \cdots, d_m$ of B with ker$\theta \cap \sum_{p=1}^m Ad_p \neq 0$ and such that all nonzero elements in this intersection have all their A-coefficients being nonzero. Set

$$\mathcal{K} = \mathrm{span}\{a_1 \in A \mid v = \sum_{p=1}^m a_p d_p \in \ker\theta \text{ for some } a_p \in A\}. \tag{12}$$

Then $\mathcal{K}$ is a nonzero subspace of A. For any $\partial \in D$, using (8) and the fact that $\ker\theta$ is a Lie ideal of $A[D]$, we have

$$\sum_{p=1}^{m} \partial(a_p)d_p = [\partial, v] \in \ker\theta, \tag{13}$$

where v is written as in (12). Thus by definition (12), $\partial(a_1) \in \mathcal{K}$, i.e., $\mathcal{K}$ is a D-stable subspace. Obviously, $A \cdot \ker\theta \subset \ker\theta$, and so $Aa_1 \subset \mathcal{K}$, i.e., $\mathcal{K}$ is a D-stable ideal of $(A, \cdot)$. Thus $\mathcal{K} = A$. In particular $1 \in \mathcal{K}$ and so there exists some $w = d_1 + \sum_{p=2}^{m} a_p d_p \in \ker\theta$. Then for all $\partial \in D$, we have $\sum_{p=2}^{m} \partial(a_p)d_p = [\partial, w] \in \ker\theta$ with support size $\leq m-1$. This means that $\partial(a_p) = 0$ for all $p = 2, \cdots, m$ and all $\partial \in D$. In other word, $a_p \in \mathbb{F}_1$ and so $0 \neq w \in \mathbb{F}_1[D] \cap \ker\theta$, but by assumption, $\mathbb{F}_1[D]$ acts faithfully on A, a contradiction. Thus Claim 1 follows.

Now suppose $\mathcal{L}$ is a Lie ideal of $A[D]$ with $\mathcal{L} \supset \mathbb{F}_1$ and $\mathcal{L} \neq \mathbb{F}_1$. Let $n \geq 0$ be minimal so that there exists $u \in \mathcal{L} \backslash \mathbb{F}_1$ with $lev(u) = n$. Fix such u.

Claim 2. $n = 0$, i.e., $u \in (A \cap \mathcal{L}) \backslash \mathbb{F}_1$.

Suppose conversely $n > 0$. For any $y \in A[D]$, we can decompose y as $y = y^* + y_0$ such that $y_0 \in A$ and all terms in y^* have degree $\neq 0$. So, we can write $u = u^* + u_0$ and $u^* \neq 0$. Since $A[D]$ acts faithfully on A, there exists $a \in A$ with $u^*(a) \neq 0$. Set $y = [u, a] \in \mathcal{L}$. Using (8), one can calculate that $y_0 = u^*(a) \neq 0$. Thus $[u, a] \neq 0$. Using (8) again and by induction on $lev(u)$, we see that $y \in \mathcal{L}$ has leading level $lev(y) \leq n-1$ since $a \in A$. By the minimal choice of n, we must have $y = [u, a] = y_0 \in \mathbb{F}_1 \subset A$. We have $A \neq \mathbb{F}_1 a + \mathbb{F}_1$, otherwise $\mathbb{F}[D]$ does not acts faithfully on A. Choose $b \in A \backslash (\mathbb{F}_1 a + \mathbb{F}_1)$. Then by replacing a by b in the above discussion, we must also have $z_0 = [u, b] \in \mathbb{F}_1$. But then

$$w = [u, ab] = [u, a]b + a[u, b] = y_0 b + z_0 a \in \mathcal{L}. \tag{14}$$

Since $y_0 \neq 0$ and $1, a, b$ are $\mathbb{F}_1$-linearly independent by the choice of b, thus $w \notin \mathbb{F}_1$. But $w \in A$ and so $lev(w) = 0 < n$, this contradicts the minimal choice of n. Thus Claim 2 follows.

Claim 3. $A \subset \mathcal{L}$.

Let u be as in Claim 2. Since $u \notin \mathbb{F}_1$, there exists $\partial_i, i \in I$ with $\partial_i(u) \neq 0$. Then for any $x \in A$, we have $x\partial_i(u) = [x\partial_i, u] \in \mathcal{L}$, i.e., $A\partial_i(u) \subset A \cap \mathcal{L}$. Obviously, $A\partial_i(u)$ is an ideal of $(A, \cdot)$, let $\mathcal{M} \subset A \cap \mathcal{L}$ be the maximal ideal of $(A, \cdot)$ containing $A\partial_i(u)$. Then $\mathcal{M} \neq 0$, and for any $\partial \in D, x \in \mathcal{M}$, $\partial(x) = [\partial, x] \in \mathcal{L}$, thus $\mathcal{M} + \partial(\mathcal{M}) \subset \mathcal{L}$. But for any $x, y \in \mathcal{M}, a \in A$, we have $a \cdot (x + \partial(y)) = ax - \partial(a)y + \partial(ay) \in \mathcal{M} + \partial(\mathcal{M})$, i.e., $\mathcal{M} + \partial(\mathcal{M})$ is an ideal of $(A, \cdot)$. Thus $\mathcal{M} + \partial(\mathcal{M}) \subset \mathcal{M}$ by the maximality of $\mathcal{M}$, and so $\mathcal{M}$ is a D-stable ideal of $(A, \cdot)$. But A is D-simple and $\mathcal{M} \neq 0$, we have $\mathcal{M} = A$. Therefore $A \subset \mathcal{L}$.

Claim 4. If char $\mathbb{F} = 0$, then $\mathcal{L} = A[D]$.

For a given $\alpha \in J$ with $|\alpha| > 0$, note that supp(α) (cf. (7)) is finite. Using induction on the leading degree, we can suppose that we have proved that $x \in \mathcal{L}$ for all monomial $x \in A[D]$ with $deg(x) < \alpha$ and supp$(deg(x)) \subseteq$ supp(α) (note that $\{\beta \in J \,|\, \beta < \alpha$ and supp$(\beta) \subseteq$ supp$(\alpha)\}$ is a finite set, therefore we can make such an inductive assumption). For any element $\gamma \in J$, we denote the largest index i with $\gamma_i \neq 0$ by $\imath(\gamma)$. If $\gamma = 0$, we set $\imath(\gamma) = -\infty$. For any $k \in I$, we denote $\delta^{(k)} \in J$ to be the element such that $\delta^{(k)}_i = \delta_{i,k}$ for all $i \in I$. Let $j = \imath(\alpha)$, then we

can write

$$\alpha = \beta + \gamma \text{ and } \partial^{(\alpha)} = \partial^{(\gamma)}\partial^{(\beta)} = \partial_j^{\alpha_j}\partial^{(\beta)}, \tag{15}$$

where $\gamma = \alpha_j \delta^{(j)}$ such that $\imath(\beta) < j$ and $\mathrm{supp}(\beta) \subset \mathrm{supp}(\alpha)$ and $\mathrm{supp}(\beta) \neq \mathrm{supp}(\alpha)$. Choose $v \in A$ with $\partial_j(v) \neq 0$. Then we have

$$\begin{aligned}
[x\partial^{(\gamma+\delta^{(j)})}, v\partial^{(\beta)}] &= [x, v\partial^{(\beta)}]\partial^{(\gamma+\delta^{(j)})} + x[\partial^{(\gamma+\delta^{(j)})}, v\partial^{(\beta)}] \\
&= v[x, \partial^{(\beta)}]\partial^{(\gamma+\delta^{(j)})} + x[\partial_j^{\alpha_j+1}, v]\partial^{(\beta)} \\
&= v[x, \partial^{(\beta)}]\partial^{(\gamma+\delta^{(j)})} + x\sum_{s=1}^{\alpha_j+1} \binom{\alpha_j+1}{s}\partial_j^s(v)\partial_j^{\alpha_j+1-s}\partial^{(\beta)}.
\end{aligned} \tag{16}$$

Observe that all terms in the summand of the right-hand side except the term corresponding to $s = 1$ are all in $\mathcal{L}$ by inductive assumption since they have degree $\gamma+(1-s)\delta^{(j)}+\beta < \alpha$ and their supports are $\subseteq \mathrm{supp}(\alpha)$. Also the left-hand side is in $\mathcal{L}$ since $\beta < \alpha$ and $\mathrm{supp}(\beta) \subset \mathrm{supp}(\alpha)$. As for the first term of the right-hand side, its leading degree must be $< \alpha$ because, either $[x, \partial^{(\beta)}] = 0$ if $\beta = 0$, or else, $[x, \partial^{(\beta)}]$ is in the subspace spanned by $A\partial^{(\tau)}$ for all $\tau \in J$ with $\mathrm{supp}(\tau) \subset \mathrm{supp}(\beta)$ and $\tau \leq \beta - \delta^{(k)}$ for some $k \geq \imath(\beta)$, and for all such τ, one has $\tau + \gamma + \delta^{(j)} \leq \beta - \delta^{(k)} + \gamma + \delta^{(j)} < \beta + \gamma = \alpha$ since $k \leq \imath(\beta) < j$. Thus the first term of the right-hand side is in $\mathcal{L}$ by inductive assumption. This shows that $x\partial_j(u)\partial^{(\alpha)} \in \mathcal{L}$. Let $\mathcal{M} \subset \{y \in A \,|\, y\partial^{(\alpha)} \in \mathcal{L}\}$ be the maximal ideal of $(A, \cdot)$ containing the ideal $A\partial_j(u)\partial^{(\alpha)}$ of $(A, \cdot)$, then as in the proof of Claim 3, $\mathcal{M} = A$, i.e., $A\partial^{(\alpha)} \subset \mathcal{L}$. This shows that all $x \in A[D]$ with degree α is in $\mathcal{L}$. This completes the proof of our fourth claim, and thus the theorem follows in this case.

Now suppose $\mathrm{char}\, \mathbb{F} = p > 0$. Observe that for any $\partial \in \mathrm{Der}_{\mathbb{F}}(A)$, one has $\partial^p \in \mathrm{Der}_{\mathbb{F}}(A)$. Thus, for any ∂_i in (1), $\partial_i^{p^s}, s = 1, 2, \cdots$ are all derivations of A, and they are A-linearly independent derivations of A because of the faithful action of $A[D]$ on A. In particular, there are infinitely many A-linearly independent derivations of A in $\mathbb{F}[D]$.

Claim 5. $A\partial \subset \mathcal{L}$ for all $\partial \in \mathrm{Der}_{\mathbb{F}}(A) \cap \mathbb{F}[D]$.

Take $\partial' \in \mathrm{Der}_{\mathbb{F}}(A) \cap \mathbb{F}[D]$ such that ∂, ∂' are A-linearly independent. By Claim 3, we have

$$[x\partial\partial', a] - x\partial(\partial'(a)) = x(\partial(a)\partial' + \partial'(a)\partial) \in \mathcal{L}, \ \forall\, x, a \in A. \tag{17}$$

Replacing x by $x\partial(b)$ we obtain $x\partial(b)(\partial(a)\partial' + \partial'(a)\partial) \in \mathcal{L}$, interchanging a and b and subtracting the two expression, we conclude that $x(\partial(b)\partial'(a) - \partial(a)\partial'(b))\partial \in \mathcal{L}$. Since $A[D]$ acts faithfully on A, we can choose $a \in A$ with $\partial'(a) \neq 0$, then $\partial'(a)\partial - \partial(a)\partial' \neq 0$ since ∂, ∂' are A-linearly independent, and so, there exists $b \in A$ with $u = \partial(b)\partial'(a) - \partial(a)\partial'(b) \neq 0$. Thus $Au\partial \subset \mathcal{L}$. Let $\mathcal{M} \subset A$ be the maximal ideal of $(A, \cdot)$ containing Au, then as in the proof of Claim 3, we have $\mathcal{M} = A$. Thus $A\partial \subset \mathcal{L}$, and the claim follows.

For any $j \in \mathbb{Z}_+$, we can write $j = \sum_{s\geq 0} j_s p^s$ with $0 \leq j_s < p$, and so for any $\partial \in D$, $\partial^j = \prod_{s\geq 0}(\partial^{p^s})^{j_s}$, where all ∂^{p^s} are derivations of A. Thus, for any $\partial^{(\alpha)} \in B$, we can rewrite $\partial^{(\alpha)}$ as

$$\partial^{(\alpha)} = \prod_{s=1}^{n} d_s^{m_s}, \ 1 \leq m_s \leq p - 1. \tag{18}$$

where $d_s \in \mathbb{F}[D] \cap \mathrm{Der}_{\mathbb{F}}(A), 1 \leq s \leq n$ are some A-linearly independent derivations.

Claim 6. $A\partial^{(\alpha)} \subset \mathcal{L}$.

We shall prove this by induction on $N_\alpha = \sum_{s=1}^n m_s$ (there may be more than one way to write $\partial^{(\alpha)}$ as in (18), and so N_α may not be uniquely defined. But from the following proof one can see that this will not affect our inductive step). By Claim 3 and Claim 5, we have the result in Claim 6 if $N_\alpha \leq 1$. So suppose $N_\alpha \geq 2$. By inductive assumption, we can suppose

$$A\partial^{(\beta)} \subset \mathcal{L} \text{ for all } \partial^{(\beta)} \text{ with } N_\beta < N_\alpha. \tag{19}$$

Choose $d_{n+1} \in \mathbb{F}[D] \cap \mathrm{Der}_{\mathbb{F}}(A)$ such that $d_1, \cdots, d_{n+1}$ are A-linearly independent. Set $m_{n+1} = 1$. As in (17), for any $a, x \in A$, we have $[x\partial^{(\alpha)}d_{n+1}, a] \in \mathcal{L}$ and

$$\begin{aligned}
[xd_{n+1}\partial^{(\alpha)}, a] &= x[d_{n+1}, a]\partial^{(\alpha)} + xd_{n+1}[\partial^{(\alpha)}, a] \\
&\equiv xd_{n+1}(a)\partial^{(\alpha)} + xd_{n+1}\sum_{s=1}^{n} m_s d_s(a) \prod_{r\neq s, 1\leq r\leq n} d_r^{m_r} d_s^{m_s-1} \pmod{\mathcal{L}} \\
&\equiv x\sum_{s=1}^{n+1} m_s d_s(a) \prod_{r\neq s, 1\leq r\leq n+1} d_r^{m_r} d_s^{m_s-1} \pmod{\mathcal{L}}, \ \forall\, x, a \in A,
\end{aligned} \tag{20}$$

where the second, third equalities follow from the inductive assumption (19). Since $d_1, \cdots,$ d_{n+1} are A-linearly independent and $A[D]$ acts faithfully on A, using induction on n, there exist $a_1, \cdots, a_{n+1} \in A$ such that

$$\det(d_s(a_r)) = \begin{vmatrix} d_1(a_1) & \cdots & d_1(a_{n+1}) \\ d_2(a_1) & \cdots & d_2(a_{n+1}) \\ \cdots & \cdots & \cdots \\ d_{n+1}(a_1) & \cdots & d_{n+1}(a_{n+1}) \end{vmatrix} \in A\backslash\{0\}. \tag{21}$$

In (20), if we take a to be $a_1, \cdots, a_{n+1}$, and denote $b_s = [x\partial^{(\alpha)}d_{n+1}, a_s] \in \mathcal{L}$ and denote $y_s = \prod_{r\neq s, 1\leq r\leq n+1} d_r^{m_r} d_s^{m_s-1}$ for $s = 1, \cdots, n+1$, then we obtain equations on y_s:

$$\sum_{s=1}^{n+1} m_s x d_s(a) y_s \equiv b_s \in \mathcal{L},\ s = 1, \cdots, n+1. \tag{22}$$

Note that by (18), $1 \leq m_s \leq p-1$, thus by (21), we can solve that there exist some $u_s \in A\backslash\{0\}$ such that $xu_s y_s \in \mathcal{L}$, $1 \leq s \leq n+1$. In particular, by taking $s = n+1$, since $y_{n+1} = \partial^{(\alpha)}$, we have $xu_{n+1}\partial^{(\alpha)} \in \mathcal{L}$ for all $x \in A$, i.e., $Au_{n+1}\partial^{(\alpha)} \in \mathcal{L}$. Hence, as in the proof of Claim 5, we have $A\partial^{(\alpha)} \subset \mathcal{L}$. This proves the claim, and thus Theorem 1.1 follows. ∎

Theorem 1.2 The associative algebra $A[D]$ is simple if and only if A is D-simple and $\mathbb{F}_1[D]$ acts faithfully on A.

Proof. "⇒": Since θ is a homomorphism of associative algebras, $\ker\theta$ is an ideal of $A[D]$. Since $A[D]$ is simple, if $\ker\theta = A[D]$, then $A[D](A) = 0$, and in particular $D(A) = 0$, a

contradiction with that D is a nonzero subspace of $\mathrm{Der}_{\mathbb{F}}(A)$. Thus $\ker\theta = 0$, i.e., $A[D]$ acts faithfully on A, and so $\mathbb{F}_1[D]$ acts faithfully on A.

If $\mathcal{I}$ is a D-stable ideal of A, then clearly $\mathcal{I}[D]$ is a ideal of $A[D]$. If $\mathcal{I} \neq 0$, then $\mathcal{I}[D] = A[D]$, and thus $\mathcal{I}$ must be A. So A is D-simple.

"$\Leftarrow$": Suppose $\mathcal{I}$ is a nonzero ideal of the associative algebra $A[D]$. Then $\mathcal{I}$ is an ideal of the Lie algebra $A[D]$. From Theorem 1.1, we know that the Lie algebra $\overline{A}[D]$ is simple. Then $\mathcal{I} = A[D]$ or $\mathcal{I} \subset \mathbb{F}_1$. If $\mathcal{I} \subset \mathbb{F}_1$, since $\mathbb{F}_1$ is a field, we must have $\mathcal{I} = A[D]$. Thus, in any case, $\mathcal{I} = A[D]$. So $A[D]$ is a simple associative algebra. ∎

2 Applications

Using Theorems 1.1, 1.2, one can obtain some interesting examples of simple (associative or Lie) algebras of Weyl type. Some are well-known examples, some seem to be new.

Corollary 2.1 Let $A = \mathbb{C}[t]$ or $\mathbb{C}[t^{\pm}]$ be the polynomial ring or Laurent polynomial ring, $D = \mathrm{span}\{\frac{\partial}{\partial t}\}$, then $\overline{A}[D] = \mathbb{C}[t, \frac{\partial}{\partial t}]/\mathbb{C}$ and $\mathbb{C}[t^{\pm 1}, \frac{\partial}{\partial t}]/\mathbb{C}$, are the well-known Lie algebras of differential operators. More generally,

$$\mathbb{C}[t_1, \cdots, t_n, \frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}]/\mathbb{C} \text{ or } \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}, \frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}]/\mathbb{C}, \tag{23}$$

are simple Lie algebras. In the first case of (23), D is a subspace of locally nilpotent derivations, while in the second case, $\frac{\partial}{\partial t_i}$ can be replaced by $t_i\frac{\partial}{\partial t_i}$ so that D is a subspace of semi-simple derivations. The associative algebra $\mathbb{C}[t_1, \cdots, t_n, \frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}]$ is the rank n Weyl algebra A_n. ∎

Corollary 2.2 Let $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{Z}_+$ with $\ell = \ell_1 + \ell_2 + \ell_3 + \ell_4 \geq 1$. Let

$$A = \mathbb{C}[t_1, \cdots, t_{\ell_1+\ell_2}, t^{\pm 1}_{\ell_1+\ell_2+1}, \cdots, t^{\pm 1}_{\ell_1+\ell_2+\ell_3}, x^{\pm 1}_{\ell_1+1}, \cdots, x^{\pm 1}_{\ell}], \tag{24}$$

and let

$$D = \{\partial_i = \frac{\partial}{\partial t_i}, \partial_j = \frac{\partial}{\partial t_j} + x_j\frac{\partial}{\partial x_j}, \partial_k = x_k\frac{\partial}{\partial x_k} \mid 1 \leq i \leq \ell_1 < j \leq \ell_1+\ell_2+\ell_3 < k \leq \ell\}, \tag{25}$$

then we obtain nongraded nonlinear simple Lie algebras of Weyl type

$$\mathbb{C}[t_1, \cdots, t_{\ell_1+\ell_2}, t^{\pm 1}_{\ell_1+\ell_2+1}, \cdots, t^{\pm 1}_{\ell_1+\ell_2+\ell_3}, x^{\pm 1}_{\ell_1+1}, \cdots, x^{\pm 1}_{\ell}, \partial_1, \cdots, \partial_\ell]/\mathbb{C}, \tag{26}$$

and simple associative algebras of Weyl type

$$\mathbb{C}[t_1, \cdots, t_{\ell_1+\ell_2}, t^{\pm 1}_{\ell_1+\ell_2+1}, \cdots, t^{\pm 1}_{\ell_1+\ell_2+\ell_3}, x^{\pm 1}_{\ell_1+1}, \cdots, x^{\pm 1}_{\ell}, \partial_1, \cdots, \partial_\ell]. \tag{27}$$

Note that ∂_i are locally nilpotent, locally finite, not locally finite or semi-simple if $1 \leq i \leq \ell_1$, $\ell_1 < i \leq \ell_1 + \ell_2$, $\ell_1 + \ell_2 < i \leq \ell_1 + \ell_2 + \ell_3$ or $\ell_1 + \ell_2 + \ell_3 < i \leq \ell$. These Lie algebras seem to be new. ∎

Corollary 2.3 Let $I\!\!F$ be a field of arbitrary characteristic. Let $x_1, x_2, \cdots$ be infinite number of algebraically independent elements over $I\!\!F$. Let $A = I\!\!F(x_1, x_2, \cdots)$ be an extension field of $I\!\!F$, and let $\partial \in \mathrm{Der}_{I\!\!F}(I\!\!F(x_1, x_2, \cdots))$ such that $\partial(x_i) = x_{i+1}$ for $i \geq 1$. Set $D = I\!\!F\partial$. Then A is D-simple and $A[D]$ acts faithfully on A. Thus we obtain the simple Lie algebra $I\!\!F(x_1, x_2, \cdots)[\partial]/I\!\!F_1$ and the simple associative algebra $I\!\!F(x_1, x_2, \cdots)[\partial]$, where $I\!\!F_1 = I\!\!F$ if $\mathrm{char}\, I\!\!F = 0$ or $I\!\!F_1 = F(x_1^p, x_2^p, \cdots)$ if $\mathrm{char}\, I\!\!F = p > 0$. If $\mathrm{char}\, I\!\!F = 2$, this gives an example that $A[D]/I\!\!F_1$ is simple, but the Lie algebra AD considered in [9] is not simple. ∎

Corollary 2.4 Let $I\!\!F$ be a field of characteristic zero. Let Γ be a multiplicative abelian group and let $A = I\!\!F[\Gamma]$ be the group algebra. Since any $\lambda \in \mathrm{Hom}(\Gamma, I\!\!F^+)$ (where $I\!\!F^+$ is the additive group $I\!\!F$) gives rise a derivation $\partial_\lambda : \sum_{\alpha\in\Gamma} f_\alpha \alpha \mapsto \sum_{\alpha\in\Gamma} f_\alpha \lambda(\alpha)\alpha$, if we take a $I\!\!F$-subspace Δ of $\mathrm{Hom}(\Gamma, I\!\!F^+)$, then we have the commutative subalgebra $D = \{\partial_\lambda \,|\, \lambda \in \Delta\}$ of derivations. If $\Gamma^\Delta = \{\alpha \in \Gamma \,|\, \lambda(\alpha) = 0, \forall\, \lambda \in \Delta\} = \{1\}$, then A is D-simple and one can prove that $A[D]$ acts faithfully on A, and so we obtain the simple Lie algebra $I\!\!F[\Gamma, \Delta]/I\!\!F$ and the simple associative algebra $I\!\!F[\Gamma, \Delta]$. ∎

We would like to conclude our paper with some problems.

Problems. 1. When $A[D] \cong A'[D']$ as Lie or associative algebras?

2. What is $\mathrm{Der}_{I\!\!F}(A[D])$?

3. What is $H^2(A[D], I\!\!F)$?

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